

ON PROLONGATIONS OF SECOND-ORDER REGULAR OVERDETERMINED SYSTEMS WITH TWO INDEPENDENT AND ONE DEPENDENT VARIABLES.

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ABSTRACT. The purpose of this present paper is to investigate the geometric structure of regular overdetermined systems of second order with two independent and one dependent variables from the point of view of rank 2 prolongations. Utilizing this notion of prolongations, we characterize the type of these overdetermined systems. We also give a systematic method for constructing the geometric singular solutions by analyzing of a decomposition of this prolongation. As an applications, we determine the geometric singular solutions of Cartan's overdetermined systems.

1. INTRODUCTION

The subject of this paper is second-order regular overdetermined systems with 2 independent and 1 dependent variables. For these overdetermined systems, various pioneering works are given by many researchers (cf. [2], [9], [6], [21], [26]). In particular, the study of overdetermined involutive systems has accomplished many significant results. First E. Cartan [2] characterized overdetermined involutive systems by the condition that these admit a one-dimensional Cauchy characteristic system. He also found out the systematic method for constructing regular solutions of involutive systems. The precise definition of regular solutions is given in Definition 5.1 of this paper. Recently, these consideration are reformulated as the theory of PD-manifolds by Yamaguchi (cf. [21], [26]). In addition, Kakie (cf. [4], [5]) studied involutive systems including the existence of regular solutions in C^∞ -category and Cauchy problems by using the theory of characteristic systems.

In this paper, we investigate regular overdetermined systems from the point of view of rank 2 prolongations. Now we introduce the notion of regular overdetermined systems. Let $J^2(\mathbb{R}^2, \mathbb{R})$ be the 2-jet space:

$$(1) \quad J^2(\mathbb{R}^2, \mathbb{R}) := \{(x, y, z, p, q, r, s, t)\}$$

This space has the canonical system $C^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ given by the annihilators:

$$\varpi_0 := dz - p dx - q dy, \quad \varpi_1 := dp - r dx - s dy, \quad \varpi_2 := dq - s dx - t dy.$$

This jet space is also constructed geometrically as the Lagrange-Grassmann bundle over the standard contact 5-dimensional manifold. For more details, see [26]. On the 2-jet

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space, we consider overdetermined systems of the form:

$$(2) \quad F(x, y, z, p, q, r, s, t) = G(x, y, z, p, q, r, s, t) = 0,$$

where F and G are smooth functions on $J^2(\mathbb{R}^2, \mathbb{R})$. We set $R = \{F = G = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R})$ and restrict the canonical differential system C^2 to R . We denote it by $D(= C^2|_R)$. An overdetermined system R is called *regular* if two vectors (F_r, F_s, F_t) and (G_r, G_s, G_t) are linearly independent on R . Then, R is a submanifold of codimension 2, and the restriction $\pi_1^2|_R : R \rightarrow J^1(\mathbb{R}^2, \mathbb{R})$ of the natural projection $\pi_1^2 : J^2(\mathbb{R}^2, \mathbb{R}) \rightarrow J^1(\mathbb{R}^2, \mathbb{R})$ is a submersion. Due to the property, restricted 1-forms $\varpi_i|_R$ on R are linearly independent. Hence $D = \{\varpi_0|_R = \varpi_1|_R = \varpi_2|_R = 0\}$ is a rank 3 system on R . For brevity, we denote each restricted generator 1-form $\varpi_i|_R$ of D by ϖ_i in the following. Regular overdetermined systems or associated differential systems are classified into 4 types consisting of involutive-type, two finite-types, and torsion-type under the structure equations or the symbol algebras (see, section 3). We thus investigate regular overdetermined systems for each case by using the theory of rank 2 prolongations. Here, the notion of the rank 2 prolongations can be regraded as a generalization of prolongations with the transversality condition and some researchers introduced this notion in each way (cf. [6], [14]). In particular, this notion can be regarded as a higher-rank version of Cartan prolongation which is investigated by Montgomery and Zhitomirskii [12]. The notion of prolongations with the transversality condition is studied deeply and their structures are well-known (cf. [2], [7], [8], [21], [26]). On the other hand, there are unexplored territories for these rank 2 prolongations. Thus, one of our main motivation in this paper is to reveal the difference between prolongations with the transversality condition and rank 2 prolongations. For more details, see section 3.

The paper is organized as follows: In section 2, we prepare some terminology and notation for the study of differential systems. In section 3, we define the rank 2 prolongations of differential systems and determine the topology of fibers of regular overdetermined systems in terms of this notion (**Theorem 3.9**). As a direct consequence followed by this characterization, we obtain the specific difference between prolongations with the transversality condition and rank 2 prolongations (**Corollary 3.10**). From this characterization, we get a new point of view of the geometric structure of overdetermined systems. In section 4, we study the structures of the canonical systems \hat{D} on the rank 2 prolongations $\Sigma(R)$ of (locally) involutive systems. More precisely, we clarify the structure of nilpotent graded Lie algebras (symbol algebras) of the canonical systems on the rank 2 prolongations by using some decomposition (**Proposition 4.1**). Here, it is well-known that the symbol algebras are fundamental invariants of differential systems or filtered manifolds. We also have the tower structure of these involutive systems by the successive prolongations (see **Theorem 4.2**). In section 5, we provide two systematic approach to construct the geometric singular solutions of involutive systems. Moreover, we apply

these methods to Cartan's overdetermined system. For this system, E. Cartan [2] already gave the explicit integral representation of the regular solutions. On the other hand, the explicit description of singular solutions has not been given yet. Thus, we give the explicit integral representation of geometric singular solutions of this system.

2. REGULARITY AND SYMBOL ALGEBRA OF DIFFERENTIAL SYSTEMS

In this section, we prepare some terminology and notation for the study of differential systems. For more details, we refer the reader to [19] and [23].

2.1. Derived system, Weak derived system. Let D be a differential system on a manifold R . We denote by $\mathcal{D} = \Gamma(D)$ the sheaf of sections to D . The derived system ∂D of a differential system D is defined, in terms of sections, by $\partial \mathcal{D} := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$. In general, ∂D is obtained as a subsheaf of the tangent sheaf of R . Moreover, higher derived systems $\partial^k D$ are defined successively by $\partial^k \mathcal{D} := \partial(\partial^{k-1} \mathcal{D})$, where we set $\partial^0 D = D$ by convention. On the other hand, the k -th weak derived systems $\partial^{(k)} D$ of D are defined inductively by $\partial^{(k)} \mathcal{D} := \partial^{(k-1)} \mathcal{D} + [\mathcal{D}, \partial^{(k-1)} \mathcal{D}]$.

Definition 2.1. A differential system D is called regular (resp. weakly regular) if $\partial^k D$ (resp. $\partial^{(k)} D$) is a subbundle for each k .

If D is not weakly regular around $x \in R$, then x is called a *singular point* in the sense of Tanaka theory. These derived systems are also interpreted by using annihilators as follows [17]: Let $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$ be a differential system on R . We denote by D^\perp the annihilator subbundle of D in T^*R , that is,

$$\begin{aligned} D^\perp(x) &:= \{\omega \in T_x^*R \mid \omega(X) = 0 \text{ for any } X \in D(x)\}, \\ &= \langle \varpi_1, \dots, \varpi_s \rangle. \end{aligned}$$

Then the annihilator $(\partial D)^\perp$ of the first derived system of D is given by

$$(\partial D)^\perp = \{\varpi \in D^\perp \mid d\varpi \equiv 0 \pmod{D^\perp}\}.$$

Moreover the annihilator $(\partial^{(k+1)} D)^\perp$ of the $(k+1)$ -th weak derived system of D is given by

$$\begin{aligned} (\partial^{(k+1)} D)^\perp &= \{\varpi \in (\partial^{(k)} D)^\perp \mid d\varpi \equiv 0 \pmod{(\partial^{(k)} D)^\perp}, \\ &\quad (\partial^{(p)} D)^\perp \wedge (\partial^{(q)} D)^\perp, \ 2 \leq p, q \leq k-1\}. \end{aligned}$$

We set $D^{-1} := D$, $D^{-k} := \partial^{(k-1)} D$ ($k \geq 2$), for a weakly regular differential system D . Then we have ([19, Proposition 1.1])

(T1) There exists a unique positive integer μ such that

$$D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots$$

(T2) $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ for all $p, q < 0$.

2.2. Symbol algebra of differential system. Let (R, D) be a weakly regular differential system such that

$$TR = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} =: D.$$

For all $x \in R$, we set $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$, $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$, ($p = -2, -3, \dots, -\mu$) and

$$\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Then $\dim \mathfrak{m}(x) = \dim R$ holds. We set $\mathfrak{g}_p(x) = \{0\}$ when $p \leq -\mu-1$. For $X \in \mathfrak{g}_p(x)$, $Y \in \mathfrak{g}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined as follows: Let $\tilde{X} \in \mathcal{D}^p$, $\tilde{Y} \in \mathcal{D}^q$ be extensions ($\tilde{X}_x = X$, $\tilde{Y}_x = Y$). Then $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$, and we set $[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{g}_{p+q}(x)$. It does not depend on the choice of the extensions because of the equation

$$[f\tilde{X}, g\tilde{Y}] = fg[\tilde{X}, \tilde{Y}] + f(\tilde{X}g)\tilde{Y} - g(\tilde{Y}f)\tilde{X} \quad (f, g \in C^\infty(R)).$$

The Lie algebra $\mathfrak{m}(x)$ is a nilpotent graded Lie algebra. We call $(\mathfrak{m}(x), [\cdot, \cdot])$ the *symbol algebra* of (R, D) at x . Note that the symbol algebra $(\mathfrak{m}(x), [\cdot, \cdot])$ satisfies the generating conditions

$$[\mathfrak{g}_p, \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1} \quad (p < 0).$$

Later, Morimoto [11] introduced the notion of a filtered manifold as generalization of the weakly regular differential system.

We define a filtered manifold (R, F) by a pair of a manifold R and a tangential filtration F . Here, a tangential filtration F on R is a sequence $\{F^p\}_{p < 0}$ of subbundles of the tangent bundle TR and the following conditions are satisfied:

- (M1) $TR = F^k = \cdots = F^{-\mu} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset F^0 = \{0\}$,
- (M2) $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q}$ for all $p, q < 0$,

where $\mathcal{F}^p = \Gamma(F^p)$ is the set of sections of F^p . Let (R, F) be a filtered manifold. For $x \in R$ we set $\mathfrak{f}_p(x) := F^p(x)/F^{p+1}(x)$, and

$$\mathfrak{f}(x) := \bigoplus_{p < 0} \mathfrak{f}_p(x).$$

For $X \in \mathfrak{f}_p(x)$, $Y \in \mathfrak{f}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{f}_{p+q}(x)$ is defined as follows: Let $\tilde{X} \in \mathcal{F}^p$, $\tilde{Y} \in \mathcal{F}^q$ be extensions ($\tilde{X}_x = X$, $\tilde{Y}_x = Y$). Then $[\tilde{X}, \tilde{Y}] \in \mathcal{F}^{p+q}$, and we set $[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{f}_{p+q}(x)$. It does not depend on the choice of the extensions. The Lie algebra $\mathfrak{f}(x)$ is also a nilpotent graded Lie algebra. We call $(\mathfrak{f}(x), [\cdot, \cdot])$ the *symbol algebra* of (R, F) at x . In general it does not satisfy the generating conditions.

3. RANK 2 PROLONGATIONS OF REGULAR OVERDETERMINED SYSTEMS

In this section, we provide the rank 2 prolongations for regular overdetermined systems of second-order of codimension 2 with 2 independent and 1 dependent variables. First, we introduce the notion of the rank 2 prolongations of differential systems, in general.

Definition 3.1. Let (R, D) be a differential system. Then, the *rank 2 prolongation* of (R, D) is defined by

$$(3) \quad \Sigma(R) := \bigcup_{x \in R} \Sigma_x,$$

where $\Sigma_x = \{v \subset T_x R \mid v \text{ is a 2-dim. integral element of } D(x) \text{ (i.e. } d\varpi_i|_v = 0) \}$. Let $p : \Sigma(R) \rightarrow R$ be the projection. We define the canonical system \hat{D} on $\Sigma(R)$ by

$$(4) \quad \begin{aligned} \hat{D}(u) &:= p_*^{-1}(u), \\ &= \{v \in T_u(\Sigma(R)) \mid p_*(v) \in u\}, \end{aligned}$$

where $u \in \Sigma(R)$.

This space $\Sigma(R)$ is a subset of the following Grassmann bundle over R

$$(5) \quad J(D, 2) := \bigcup_{x \in R} J_x$$

where $J_x := \{v \subset T_x R \mid v \text{ is a 2-dim. subspace of } D(x)\}$. In general, the rank 2 prolongations $\Sigma(R)$ have singular points, that is $\Sigma(R)$ is not smooth. In fact, for prolongations, there also exists the notion of *prolongations with the transversality condition* :

$$(6) \quad R^{(1)} = \bigcup_{x \in R} R_x^{(1)},$$

where $R_x^{(1)} = \{2\text{-dim. integral elements of } D(x), \text{ transversal to } \text{Ker}(\pi_1^2|_{R*})\}$. For this notion, it is well-known the geometric structures by many workers ([1], [2], [21]). In this paper, we examine the prolongations except for this transversality condition.

Next, we explain a classification of the type of overdetermined systems under the structure equations or the corresponding symbols. Let (R, D) be a regular overdetermined system. If (R, D) does not have torsion, that is, the prolongation $R^{(1)}$ is onto, then the structure equation of this system is one of the following three cases ([26, the case of codim $\mathfrak{f} = 2$ of Case $n = 2$ in p. 346–347]):

- (I) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w :

$$(7) \quad \begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0, \\ d\varpi_1 &\equiv 0 && \text{mod } \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi && \text{mod } \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

(II) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w :

$$(8) \quad \begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0, \\ d\varpi_1 &\equiv && \omega_2 \wedge \pi && \text{mod } \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_1 \wedge \pi && \text{mod } \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

(III) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w :

$$(9) \quad \begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0, \\ d\varpi_1 &\equiv \omega_1 \wedge \pi && \text{mod } \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv && \omega_2 \wedge \pi && \text{mod } \varpi_0, \varpi_1, \varpi_2, \end{aligned}$$

Now we consider the case where torsion exists, that is, $R^{(1)}$ is not onto. In fact, then the structure equation (or the symbol) of torsion type has the normal form by the obtained result in [15]. This fact also follows from the technique of the proof of [13, Theorem 3.3]. Namely, if (R, D) has torsion at $w \in R$, we have the following structure equation at w .

(IV) There exists a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi\}$ around $w \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds at w :

$$(10) \quad \begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0, \\ d\varpi_1 &\equiv \omega_1 \wedge \omega_2 && \text{mod } \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv && \omega_2 \wedge \pi && \text{mod } \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

Here the types of (I), (II), (III) and (IV) correspond to differential systems of involutive type, two finite types, and torsion type respectively (cf. [25], [26]). From now on, We often call these systems the differential systems of type (k) , where $k = \text{I, II, III, IV}$.

Remark 3.2. The structures of prolongations $R^{(1)}$ with the transversality condition for R of types of (I), (II), (III) and (IV) are known. Indeed, $R^{(1)} \rightarrow R$ is a \mathbb{R} -bundle for the type of (I). Moreover $R^{(1)}$ is diffeomorphic to R for types of (II) or (III), and the set $R^{(1)}$ is empty for types of (IV).

One of the main purpose of this section is to clarify the difference between $R^{(1)}$ and $\Sigma(R)$. First, we consider the case of type (I).

Lemma 3.3. *Let (R, D) be a differential system of type (I) with 2 independent and 1 dependent variables. Then the rank 2 prolongation $\Sigma(R)$ is a smooth submanifold of $J(D, 2)$. Moreover, it is a S^1 -bundle over R .*

PROOF. Let $\pi : J(D, 2) \rightarrow R$ be the projection and U an open set in R . Then $\pi^{-1}(U)$ is covered by 3 open sets in $J(D, 2)$, that is,

$$(11) \quad \pi^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi} \cup U_{\omega_2\pi},$$

where

$$\begin{aligned} U_{\omega_1\omega_2} &:= \{v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \omega_2|_v \neq 0\}, \quad U_{\omega_1\pi} := \{v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \pi|_v \neq 0\}, \\ U_{\omega_2\pi} &:= \{v \in \pi^{-1}(U) \mid \omega_2|_v \wedge \pi|_v \neq 0\}. \end{aligned}$$

We explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}, U_{\omega_1\pi}, U_{\omega_2\pi}$. First we consider it on $U_{\omega_1\omega_2}$. For $w \in U_{\omega_1\omega_2}$, w is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting π to w , we can introduce the inhomogeneous coordinate p_i^1 ($i = 1, 2$) of fibers of $J(D, 2)$ around w with $\pi|_w = p_1^1(w)\omega_1|_w + p_2^1(w)\omega_2|_w$. Moreover, w satisfies $d\varpi_2|_w \equiv 0$ in (7). Hence, we show that

$$d\varpi_2|_w \equiv \omega_2|_w \wedge \pi|_w \equiv p_1^1(w)\omega_2|_w \wedge \omega_1|_w.$$

Thus we obtain the defining equations $f = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D, 2)$, where $f = p_1^1$, that is, $\{f = 0\} \subset U_{\omega_1\omega_2}$. Then df does not vanish on $\{f = 0\}$. In the same way, on $U_{\omega_1\pi}$, df does not vanish on $\Sigma(R)$. Finally we consider on $U_{\omega_2\pi}$. Then an element $w \in U_{\omega_2\pi}$ is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting ω_1 to w , we can introduce the inhomogeneous coordinate p_i^3 ($i=1, 2$) of fibers of $J(D, 2)$ around w with $\omega_1|_w = p_1^3(w)\omega_2|_w + p_2^3(w)\pi|_w$. Moreover, w satisfies $d\varpi_2|_w \equiv 0$. However we have

$$d\varpi_2|_w \equiv \omega_2|_w \wedge \pi|_w \neq 0.$$

Thus, there does not exist an integral element, that is, $U_{\omega_2\pi} \cap p^{-1}(U) = \emptyset$. Therefore, we conclude $\Sigma(R)$ is a submanifold in $J(D, 2)$.

Next, we show that the topology of its fibers is S^1 . For any open set $U \subset R$, we obtain the covering

$$(12) \quad p^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi}.$$

Then the canonical system \hat{D} of rank 3 is given by

$$\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_\pi = 0\} \quad \text{on } U_{\omega_1\omega_2},$$

where $\varpi_\pi = \pi - a\omega_2$ and a is fiber coordinate and

$$\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_2} = 0\} \quad \text{on } U_{\omega_1\pi},$$

where $\varpi_{\omega_2} = \omega_2 - b\pi$ and b is a fiber coordinate. To prove the statement, we consider the gluing of $(\Sigma(R), \hat{D})$. Let $w \in p^{-1}(U)$ be a point in $U_{\omega_1\pi} \subset p^{-1}(U)$. Here, if $w \notin U_{\omega_1\omega_2}$, then we have $b = 0$ because of the condition $\omega_1 \wedge \omega_2 = 0$. Thus we show that $b \neq 0$ on $U_{\omega_1\omega_2} \cap U_{\omega_1\pi}$ and $b = 0$ on $U_{\omega_1\pi} \setminus U_{\omega_1\omega_2}$. Hence we can prove that the topology of fibers is S^1 . Indeed, we obtain the transition function ϕ of $(\Sigma(R), \hat{D})$ on $U_{\omega_1\omega_2} \cap U_{\omega_1\pi}$ defined by

$$\phi(v, a) = \left(v, b := \frac{1}{a} \right) \quad \text{for } a \neq 0,$$

where v is a local coordinate on R . This map ϕ satisfies the condition $\phi_*\hat{D} = \hat{D}$ on $U_{\omega_1\omega_2} \cap U_{\omega_1\pi}$. Thus ϕ is the projective transformation of $\mathbb{RP}^1 \cong S^1$. Note that each fiber in codimension 1 submanifold $\{b = 0\} \subset U_{\omega_1\pi}$ corresponds to the point at infinity of \mathbb{RP}^1 . \square

Next, we consider the case of type (II).

Lemma 3.4. *Let (R, D) be a differential system of type (II) with 2 independent and 1 dependent variables. Then the rank 2 prolongation $\Sigma(R)$ is diffeomorphic to R .*

Remark 3.5. We emphasize that (R, D) and $(\Sigma(R), \hat{D})$ are different as differential systems. Indeed D is a rank 3 differential system on R , but \hat{D} is a rank 2 differential system on $\Sigma(R)$.

PROOF. In this situation we also use the covering (11) of $\pi^{-1}(U)$ for the Grassmann bundle $J(D, 2)$ and explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}, U_{\omega_1\pi}, U_{\omega_2\pi}$. First we consider it on $U_{\omega_1\omega_2}$. For $w \in U_{\omega_1\omega_2}$, w is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting π to w , we can introduce the inhomogeneous coordinate p_i^1 of fibers of $J(D, 2)$ around w with $\pi|_w = p_1^1(w)\omega_1|_w + p_2^1(w)\omega_2|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$ in (8). Thus we get

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_2|_w \wedge \pi|_w \equiv p_1^1(w)\omega_2|_w \wedge \omega_1|_w, \\ d\varpi_2|_w &\equiv \omega_1|_w \wedge \pi|_w \equiv p_2^1(w)\omega_1|_w \wedge \omega_2|_w. \end{aligned}$$

In this way, we obtain the defining equations $f = g = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D, 2)$, where $f = p_1^1, g = p_2^1$. Hence we have one trivial integral element. Next we consider on $U_{\omega_1\pi}$. In the same way, by restricting ω_2 to w , we can introduce the inhomogeneous coordinate p_i^2 of fibers of $J(D, 2)$ around w with $\omega_2|_w = p_1^2(w)\omega_1|_w + p_2^2(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However we have $d\varpi_2|_w \equiv \omega_1|_w \wedge \pi|_w \neq 0$. Hence there does not exist an integral element. Finally we consider on $U_{\omega_2\pi}$. In this situation, by restricting ω_1 to w , we can also introduce the inhomogeneous coordinate p_i^3 of fibers of $J(D, 2)$ around w with $\omega_1|_w = p_1^3(w)\omega_2|_w + p_2^3(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However, we have $d\varpi_1|_w \equiv \omega_2|_w \wedge \pi|_w \neq 0$. Hence there does not exist an integral element. Therefore, $\Sigma(R)$ is a section of the Grassmann bundle $J(D, 2)$ over R . \square

Next we consider the case of type (III).

Lemma 3.6. *Let (R, D) be a differential system of type (III) with 2 independent and 1 dependent variables. Then the rank 2 prolongation $\Sigma(R)$ is equal to R .*

Remark 3.7. Two differential systems (R, D) and $(\Sigma(R), \hat{D})$ are different in the same way to Remark 3.5.

PROOF. We also use the covering (11) of $\pi^{-1}(U)$ for the Grassmann bundle $J(D, 2)$ and explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}, U_{\omega_1\pi}, U_{\omega_2\pi}$. First we consider it on $U_{\omega_1\omega_2}$. For $w \in U_{\omega_1\omega_2}$, w is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence by restricting π to w , we can introduce the inhomogeneous coordinate p_i^1 of fibers of $J(D, 2)$ around w with $\pi|_w = p_1^1(w)\omega_1|_w + p_2^1(w)\omega_2|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$ in (9). Thus we have

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_1|_w \wedge \pi|_w \equiv p_2^1(w)\omega_1|_w \wedge \omega_2|_w, \\ d\varpi_2|_w &\equiv \omega_2|_w \wedge \pi|_w \equiv p_1^1(w)\omega_2|_w \wedge \omega_1|_w. \end{aligned}$$

In this way, we obtain the defining equations $f = g = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D, 2)$, where $f = p_2^1, g = p_1^1$. Hence, we have one trivial integral element. Next we consider on $U_{\omega_1\pi}$. For the same way, by restricting ω_2 to w , we can introduce the inhomogeneous coordinate p_i^2 of fibers of $J(D, 2)$ around w with $\omega_2|_w = p_1^2(w)\omega_1|_w + p_2^2(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However we have $d\varpi_1|_w \equiv \omega_1|_w \wedge \pi|_w \neq 0$. Hence there does not exist an integral element. Finally we consider on $U_{\omega_2\pi}$. In this situation, by restricting ω_1 to w , we can also introduce the inhomogeneous coordinate p_i^3 of fibers of $J(D, 2)$ around w with $\omega_1|_w = p_1^3(w)\omega_2|_w + p_2^3(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However we have $d\varpi_2|_w \equiv \omega_2|_w \wedge \pi|_w \neq 0$. Hence there does not exist an integral element. \square

Finally we consider the case of type (IV).

Lemma 3.8. *Let (R, D) be a differential system of type (IV) with 2 independent and 1 dependent variables. Then, the rank 2 prolongation $\Sigma(R)$ is equal to R .*

PROOF. We also use the covering (11) of $\pi^{-1}(U)$ for the Grassmann bundle $J(D, 2)$ and explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}, U_{\omega_1\pi}, U_{\omega_2\pi}$. First we consider it on $U_{\omega_1\omega_2}$. For $w \in U_{\omega_1\omega_2}$, w is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting π to w , we can introduce the inhomogeneous coordinate p_i^1 of fibers of $J(D, 2)$ around w with $\pi|_w = p_1^1(w)\omega_1|_w + p_2^1(w)\omega_2|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$ in (10). However we have $d\varpi_1|_w \equiv \omega_1|_w \wedge \omega_2|_w \neq 0$. Hence there does not exist an integral element. Next we consider on $U_{\omega_1\pi}$. For the same way, by restricting ω_2 to w , we can also introduce the inhomogeneous coordinate p_i^2 of fibers of $J(D, 2)$ around w with $\omega_2|_w = p_1^2(w)\omega_1|_w + p_2^2(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. Thus we obtain

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_1|_w \wedge \omega_2|_w \equiv p_2^2(w)\omega_1|_w \wedge \pi|_w, \\ d\varpi_2|_w &\equiv \omega_2|_w \wedge \pi|_w \equiv p_1^2(w)\omega_1|_w \wedge \pi|_w. \end{aligned}$$

In this way, we obtain the defining equations $f = g = 0$ of $\Sigma(R)$ in $U_{\omega_1\pi}$ of $J(D, 2)$, where $f = p_2^2, g = p_1^2$. Hence we have one trivial integral element. Finally, we consider on $U_{\omega_2\pi}$. For this situation, by restricting ω_1 to w , we can also introduce the inhomogeneous

coordinate p_i^3 of fibers of $J(D, 2)$ around w with $\omega_1|_w = p_1^3(w)\omega_2|_w + p_2^3(w)\pi|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However, we have $d\varpi_2|_w \equiv \omega_2|_w \wedge \pi|_w \neq 0$. Hence there does not exist an integral element. \square

Summarizing these lemmas in this section, we obtain the following theorem.

Theorem 3.9. *Let (R, D) be a second-order regular overdetermined system of codimension 2 for 2 independent and 1 dependent variables. Then the only involutive systems (i.e. type of (I)) have non-trivial rank 2 prolongations $\Sigma(R)$. Moreover, in this case, the rank 2 prolongation $\Sigma(R)$ is a S^1 -bundle over R .*

Corollary 3.10. *Let (R, D) be a second-order regular overdetermined system of codimension 2 for 2 independent and 1 dependent variables. Then we have*

$$\begin{aligned} R^{(1)} = \Sigma(R) &\iff \text{(II), (III),} \\ R^{(1)} \neq \Sigma(R) &\iff \text{(I), (IV).} \end{aligned}$$

4. STRUCTURES OF RANK 2 PROLONGATIONS FOR INVOLUTIVE SYSTEMS

In this section, we study the geometric structures of rank 2 prolongations $(\Sigma(R), \hat{D})$ of involutive systems (R, D) with respect to 2 independent and 1 dependent variables. For this purpose, we consider the decomposition

$$(13) \quad \Sigma(R) = \Sigma_0 \cup \Sigma_1,$$

where $\Sigma_i = \{w \in \Sigma(R) \mid \dim(w \cap \text{fiber}) = i\}$ ($i = 0, 1$). Here “fiber” means that the fiber of $TR \supset D \rightarrow TJ^1$. For the covering of the fibration $p: \Sigma(R) \rightarrow R$, we have

$$\Sigma_0|_{p^{-1}(U)} = U_{\omega_1\omega_2}, \quad \Sigma_1|_{p^{-1}(U)} = U_{\omega_1\pi} \setminus U_{\omega_1\omega_2}.$$

The set Σ_0 is an open subset in $\Sigma(R)$, and Σ_1 is a codimension 1 submanifold in $\Sigma(R)$.

Considering this decomposition, we obtain the following result.

Proposition 4.1. *For any point $w \in \Sigma_0$, the symbol algebra $\mathfrak{f}^0(w)$ is isomorphic to*

$$\mathfrak{f}^0 := \mathfrak{f}_{-4} \oplus \mathfrak{f}_{-3} \oplus \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1}$$

whose bracket relations are given by

$$[X_a, X_{\omega_2}] = X_\pi, \quad [X_\pi, X_{\omega_2}] = X_2, \quad [X_1, X_{\omega_1}] = [X_2, X_{\omega_2}] = X_0,$$

and the other brackets are trivial. Here $\{X_0, X_1, X_2, X_{\omega_1}, X_{\omega_2}, X_\pi, X_a\}$ is a basis of \mathfrak{f}^0 and

$$\mathfrak{f}_{-1} = \{X_{\omega_1}, X_{\omega_2}, X_a\}, \quad \mathfrak{f}_{-2} = \{X_\pi\}, \quad \mathfrak{f}_{-3} = \{X_1, X_2\}, \quad \mathfrak{f}_{-4} = \{X_0\}.$$

For any point $w \in \Sigma_1$, the symbol algebra $\mathfrak{f}^1(w)$ is isomorphic to

$$\mathfrak{f}^1 := \mathfrak{f}_{-4} \oplus \mathfrak{f}_{-3} \oplus \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1}$$

whose bracket relations are given by

$$[X_b, X_\pi] = X_{\omega_2}, \quad [X_\pi, X_{\omega_2}] = X_2, \quad [X_1, X_{\omega_1}] = X_0,$$

and the other brackets are trivial. Here $\{X_0, X_1, X_2, X_{\omega_1}, X_{\omega_2}, X_\pi, X_b\}$ is a basis of \mathfrak{f}^1 and

$$\mathfrak{f}_{-1} = \{X_{\omega_1}, X_\pi, X_b\}, \quad \mathfrak{f}_{-2} = \{X_{\omega_2}\}, \quad \mathfrak{f}_{-3} = \{X_1, X_2\}, \quad \mathfrak{f}_{-4} = \{X_0\}.$$

PROOF. We first prove the assertion for the symbol algebras on Σ_0 . We recall that the canonical system \hat{D} on Σ_0 is given by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_\pi = 0\}$, where $\varpi_\pi = \pi - a\omega_2$. Then the structure equation of \hat{D} on Σ_0 can be written as

$$(14) \quad \begin{aligned} d\varpi_i &\equiv 0 && \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \\ d\varpi_\pi &\equiv \omega_2 \wedge (da + f\omega_1) && \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \end{aligned}$$

where f is an appropriate function. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is equal to the structure equation (7) of (R, D) . Now we provide a filtration structure $\{F^p\}_{p=-1}^{-4}$ on $\Sigma(R)$ around $w \in \Sigma_0$. We set $F^{-4} := T\Sigma(R)$, $F^{-3} := \{\varpi_0 = 0\}$, $F^{-2} := \partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$, $F^{-1} := \hat{D}$. Moreover, for $w \in \Sigma_0$, we set $\mathfrak{f}_{-1}(w) := F^{-1}(w) = \hat{D}(w)$, $\mathfrak{f}_{-2}(w) := F^{-2}(w)/F^{-1}(w)$, $\mathfrak{f}_{-3}(w) := F^{-3}(w)/F^{-2}(w)$, $\mathfrak{f}_{-4}(w) := F^{-4}(w)/F^{-3}(w)$, and

$$\mathfrak{f}^0(w) = \mathfrak{f}_{-4}(w) \oplus \mathfrak{f}_{-3}(w) \oplus \mathfrak{f}_{-2}(w) \oplus \mathfrak{f}_{-1}(w).$$

Then, by the definition of symbol algebras associated with filtration structures in Section 2, $\mathfrak{f}^0(w)$ has the structure of a nilpotent graded Lie algebra. We consider the bracket relation of $\mathfrak{f}^0(w)$. We take a coframe around $w \in \Sigma_0$ given by

$$(15) \quad \{\varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \omega_1, \omega_2, \varpi_a := da + f\omega_1\},$$

and the dual frame

$$(16) \quad \{X_0, X_1, X_2, X_\pi, X_{\omega_1}, X_{\omega_2}, X_a\}.$$

Then, the structure equations of each subbundle in the filtration $\{F^p\}_{p=-1}^{-4}$ can be written by the coframe

$$\begin{aligned} d\varpi_i &\equiv 0 && \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \\ d\varpi_\pi &\equiv \omega_2 \wedge \varpi_a && \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_\pi, \end{aligned}$$

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 && \text{mod } \varpi_0, \\ d\varpi_1 &\equiv 0 && \text{mod } \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \varpi_\pi && \text{mod } \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

$$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \text{mod } \varpi_0, \varpi_1 \wedge \varpi_2, \varpi_1 \wedge \varpi_\pi, \varpi_2 \wedge \varpi_\pi.$$

We set

$$[X_{\omega_2}, X_a] = AX_\pi, \quad (A \in \mathbb{R}).$$

Then

$$\begin{aligned} d\varpi_\pi(X_{\omega_2}, X_a) &= X_{\omega_2}\varpi_\pi(X_a) - X_a\varpi_\pi(X_{\omega_2}) - \varpi_\pi([X_{\omega_2}, X_a]), \\ &= -\varpi_\pi([X_{\omega_2}, X_a]) = -A. \end{aligned}$$

On the other hand

$$\begin{aligned} d\varpi_\pi(X_{\omega_2}, X_a) &= \omega_2(X_{\omega_2})\varpi_a(X_a) - \varpi_a(X_{\omega_2})\omega_2(X_a), \\ &= 1. \end{aligned}$$

Therefore, $A = -1$. The other brackets are also obtained by the same argument and the definition of symbol algebras associated with the filtration structure. Thus we have the bracket relation of \mathfrak{f}^0 .

We next prove the assertion for the symbol algebras on Σ_1 . We recall that Σ_1 is locally given by $U_{\omega_1\pi} \setminus U_{\omega_1\omega_2}$. Thus we may assume on $U_{\omega_1\pi} \setminus U_{\omega_1\omega_2} = \{b = 0\} \subset U_{\omega_1\pi}$. Then the canonical system \hat{D} is given by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_2} = 0\}$, where $\varpi_{\omega_2} = \omega_2 - b\pi$. Note that $\varpi_{\omega_2} = \omega_2$ on Σ_1 . The structure equation of \hat{D} at a point $w \in \Sigma_1 = \{b = 0\}$ is

$$\begin{aligned} (17) \quad d\varpi_i &\equiv 0 \quad \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \\ d\varpi_{\omega_2} &\equiv \pi \wedge (db + f\omega_1) \quad \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \end{aligned}$$

where f is an appropriate function. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is equal to the structure equation (7) of (R, D) . Here, we take the filtration which is same to the case of \mathfrak{f}^0 . Then we have the symbol algebra $\mathfrak{f}^1(w)$ at a point $w \in \Sigma_1$ given by

$$\mathfrak{f}^1(w) = \mathfrak{f}_{-4}(w) \oplus \mathfrak{f}_{-3}(w) \oplus \mathfrak{f}_{-2}(w) \oplus \mathfrak{f}_{-1}(w).$$

We consider the bracket relation of $\mathfrak{f}^1(w)$. We take a coframe around $w \in \Sigma_1$ given by

$$(18) \quad \{\varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \omega_1, \pi, \varpi_b := db + f\omega_1\},$$

and the dual frame

$$(19) \quad \{X_0, X_1, X_2, X_{\omega_2}, X_{\omega_1}, X_\pi, X_b\}.$$

Then the structure equations of each subbundle in the filtration $\{F^p\}_{p=-1}^{-4}$ can be written by the coframe

$$\begin{aligned} d\varpi_i &\equiv 0 \quad \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \\ d\varpi_{\omega_2} &\equiv \pi \wedge \varpi_b \quad \text{mod } \varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \end{aligned}$$

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \text{mod } \varpi_0, \\ d\varpi_1 &\equiv 0 \quad \text{mod } \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi \quad \text{mod } \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

$$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 \quad \text{mod } \varpi_0, \varpi_1 \wedge \varpi_2, \varpi_1 \wedge \varpi_{\omega_2}, \varpi_2 \wedge \varpi_{\omega_2}.$$

By the definition of the symbol algebras associated with the filtration structure and same argument in the proof of \mathfrak{f}^0 , we obtain the bracket relation for \mathfrak{f}^1 . \square

In the rest of this section, we mention a tower structure constructed by successive rank 2 prolongations of involutive systems.

Theorem 4.2. *Let (R, D) be a regularly involutive system with 2 independent 1 dependent variables. Then the k -th rank 2 prolongation $(\Sigma^k(R), \hat{D}^k)$ of (R, D) is also S^1 -bundle over $\Sigma^{k-1}(R)$.*

PROOF. From the expressions (14) or (17) of the structure equations of (R, D) , we easily show that we can define the k -th rank 2 prolongation $(\Sigma^k(R), \hat{D}^k)$ successively. Then we have the assertion by using the same argument in the proof of Lemma 3.3 successively. \square

5. GEOMETRIC SINGULAR SOLUTIONS OF INVOLUTIVE SYSTEMS

In this section, we investigate the geometric singular solutions of involutive systems with 2 independent and 1 dependent variables. We first define the notion of the geometric singular solutions for regular PDEs ([13, 14]).

Definition 5.1. Let (R, D) be a second-order regular PDE in $J^2(\mathbb{R}^2, \mathbb{R})$. For a 2-dimensional integral manifold S of R , if the restriction $\pi_1^2|_R : R \rightarrow J^1(\mathbb{R}^2, \mathbb{R})$ of the natural projection $\pi_1^2 : J^2 \rightarrow J^1$ is an immersion on an open dense subset in S , then we call S a *geometric solution* of (R, D) . If all points of geometric solutions S are immersion points, then we call S *regular solutions*. On the other hand, if geometric solutions S have a nonimmersion point, then we call S *singular solutions*.

From the definition, the image $\pi_1^2(S)$ of a geometric solution S by the projection π_1^2 is Legendrian in $J^1(\mathbb{R}^2, \mathbb{R})$, $(\varpi_0|_{\pi_1^2(S)} = d\varpi_0|_{\pi_1^2(S)} = 0)$. From the proof of Lemmas 3.4 and 3.6, there does not exist singular solutions of equations of types (II) and (III). On the other hand, Lemma 3.8 says the possibility of the existence of singular solutions for torsion type (IV). Of course, there does not exist regular solutions for these equations of torsion type. From now on, we investigate only involutive systems.

Let (R, D) be a regularly involutive system of codimension 2 with 2 independent and 1 dependent variables. For this system (R, D) , we give two methods of the construction of geometric singular solutions which are given by the following.

- (i) We construct singular solutions of (R, D) by using solutions of special type of rank 2 prolongations $(\Sigma(R), \hat{D})$.
- (ii) We construct singular solutions of (R, D) in terms of solutions (integral curves) of special type of rank 2 differential system D_B on a 5-dimensional manifold B .

The approach (i) is applicable to PDEs except involutive systems [14]. On the other hand, the approach (ii) is a method specialized for involutive systems.

We first mention the principle of the approach (i). We recall the decomposition (13) of the rank 2 prolongations $(\Sigma(R), \hat{D})$ of (R, D) . Here $(U_{\omega_1\omega_2}, \hat{D})$ is the rank 2 prolongation with the independence condition $\omega_1 \wedge \omega_2 \neq 0$. In general, for given second order regular overdetermined system $R = \{F = G = 0\}$ with independent variables x, y , this prolongation corresponds to a third order PDE system which is obtained by partial derivation of $F = G = 0$ for the two variables x, y . If we construct a solution of the system $(U_{\omega_1\omega_2}, \hat{D})$, this solution S is regular by the definition of Σ_0 . On the other hand, if we construct a solution S of $(\Sigma(R), \hat{D})$ passing through $\Sigma_1 = \{b = 0\} \subset U_{\omega_1\pi}$, this solution S is a singular solution of (R, D) from the decomposition (13). Thus, our strategy of this case is to find such a solution S of $(\Sigma(R), \hat{D})$.

We next mention the principle of the approach (ii). In fact, E. Cartan [2] characterized the overdetermined involutive systems R by the condition that R admits a 1-dimensional Cauchy characteristic system. Here, the Cauchy characteristic system $Ch(D)$ of a differential system D on R is defined by

$$Ch(D)(x) := \{X \in D(x) \mid X \rfloor d\varpi_i \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2} \text{ for } i = 0, 1, 2\},$$

where, \rfloor denotes the interior product (i.e., $X \rfloor d\varpi(Y) = d\varpi(X, Y)$), and $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is defined locally by defining 1-forms $\{\varpi_0, \varpi_1, \varpi_2\}$. From the expression (7) of the structure equation of an involutive system (R, D) , we obtain $Ch(D) = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega_2 = \pi = 0\}$. This system $Ch(D)$ gives a 1-dimensional foliation. Hence, a leaf space $B := R/Ch(D)$ is locally a 5-dimensional manifold. For this fibration $\pi_B^R : R \rightarrow B$, it is well-known that there exists a rank 2 differential system D_B on the quotient space B ([2], [16], [26]). Hence, if we construct integral curves of rank 2 differential system (B, D_B) , we obtain integral surfaces S of (R, D) by using the fibration $\pi_B^R : R \rightarrow B$. Our strategy of this case is to find a singular solution among solutions obtained from such a technique. This principle is nothing but the theory of characteristic system. Namely, this approach is a theory of reduction into ordinary differential equations.

As an application of the above discussion, we construct singular solutions of a typical equation based on the above two approach in the following subsection.

5.1. Singular solutions of Cartan's overdetermined system. We consider Cartan's overdetermined system

$$R = \left\{ r = \frac{t^3}{3}, \quad s = \frac{t^2}{2} \right\}.$$

The canonical system D on R is given by

$$\varpi_0 = dz - p dx - q dy, \quad \varpi_1 = dp - \frac{t^3}{3} dx - \frac{t^2}{2} dy, \quad \varpi_2 = dq - \frac{t^2}{2} dx - t dy,$$

and the structure equation of D is given by

$$\begin{aligned} (20) \quad d\varpi_0 &\equiv dx \wedge dp + dy \wedge dq, & \text{mod } \varpi_0 \\ d\varpi_1 &\equiv -t^2 dt \wedge dx - t dt \wedge dy, & \text{mod } \varpi_0, \varpi_1, \varpi_2, \\ d\varpi_2 &\equiv -t dt \wedge dx - dt \wedge dy, & \text{mod } \varpi_0, \varpi_1, \varpi_2. \end{aligned}$$

We take a new coframe

$$\{\varpi_0, \hat{\varpi}_1 := \varpi_1 - t\varpi_2, \varpi_2, \pi := dt, \omega_1 := dx, \omega_2 := tdx + dy\}.$$

For this coframe, the above structure equation is rewritten as

$$\begin{aligned} (21) \quad d\varpi_0 &\equiv \omega_1 \wedge \hat{\varpi}_1 + \omega_2 \wedge \varpi_2, & \text{mod } \varpi_0, \\ d\hat{\varpi}_1 &\equiv 0 & \text{mod } \varpi_0, \hat{\varpi}_1, \varpi_2, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi, & \text{mod } \varpi_0, \hat{\varpi}_1, \varpi_2. \end{aligned}$$

Hence this system (R, D) is locally involutive.

We first construct singular solutions of (R, D) by using the approach (i). For this purpose, we need to prepare the rank 2 prolongation $(\Sigma(R), \hat{D})$ of (R, D) in terms of the Grassmann bundle $\pi : J(D, 2) \rightarrow R$. For any open set $U \subset R$, $\pi^{-1}(U)$ is covered by 3 open sets in $J(D, 2)$ such that $\pi^{-1}(U) = U_{xy} \cup U_{xt} \cup U_{yt}$, where

$$\begin{aligned} U_{xy} &:= \{w \in \pi^{-1}(U) \mid dx|_w \wedge dy|_w \neq 0\}, \quad U_{xt} := \{w \in \pi^{-1}(U) \mid dx|_w \wedge dt|_w \neq 0\}, \\ U_{yt} &:= \{w \in \pi^{-1}(U) \mid dy|_w \wedge dt|_w \neq 0\}. \end{aligned}$$

We next explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in U_{xy}, U_{xt}, U_{yt} . First, we consider it in U_{xy} . For $w \in U_{xy}$, w is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting dt to w , we can introduce the inhomogeneous coordinate p_i^1 of fibers of $J(D, 2)$ around w with $dt|_w = p_1^1(w)dx|_w + p_2^1(w)dy|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. Thus we have

$$\begin{aligned} d\varpi_1|_w &\equiv (t^2 p_2^1(w) - t p_1^1(w)) dx|_w \wedge dy|_w, \\ d\varpi_2|_w &\equiv (t p_2^1(w) - p_1^1(w)) dx|_w \wedge dy|_w. \end{aligned}$$

In this way, we obtain the defining equations $f = 0$ of $\Sigma(R)$ in U_{xy} of $J(D, 2)$, where $f = p_1^1 - t p_2^1$. Then df does not vanish on $\{f = 0\}$. Next we consider in U_{xt} . For $w \in U_{xt}$, w is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting dy to

w , we can introduce the inhomogeneous coordinate p_i^2 of fibers of $J(D, 2)$ around w with $dy|_w = p_1^2(w)dx|_w + p_2^2(w)dt|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. In this situation, it is sufficient to consider the condition $d\varpi_2|_w (\equiv (t + p_1^2(w))dx|_w \wedge dt|_w) \equiv 0$. Then, for the defining function $f = t + p_1^2$ of $\Sigma(R)$, df does not vanish on $\Sigma(R)$. Finally, we consider in U_{yt} . For $w \in U_{yt}$, w is a 2-dimensional subspace of $D(v)$, where $p(w) = v$. Hence, by restricting dx to w , we can introduce the inhomogeneous coordinate p_i^3 of fibers of $J(D, 2)$ around w with $dx|_w = p_1^3(w)dy|_w + p_2^3(w)dt|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. Here, $d\varpi_2|_w \equiv (1 + tp_1^3(w))dy|_w \wedge dt|_w$. Then, for the defining function $f = 1 + tp_1^3$ of $\Sigma(R)$, df does not vanish on $\Sigma(R)$. Therefore, we have the covering for the fibration $p : \Sigma(R) \rightarrow R$ such that $p^{-1}(U) = U_{xy} \cup U_{xt} \cup U_{yt}$. However this covering is not essential.

Proposition 5.2. *Let R be Cartan's overdetermined system and U an open set on R . Then we have*

$$(22) \quad p^{-1}(U) = U_{xy} \cup U_{xt}.$$

PROOF. We show that $U_{yt} \subset U_{xt}$. Let w be any point in U_{yt} . Here, if $w \notin U_{xt}$, we have

$$dx|_w \wedge dt|_w = -\frac{1}{t}(w)dy|_w \wedge dt|_w.$$

Hence we have the condition $1/t(w) = 0$. However there does not exist such a point w . \square

We have the following description of the canonical system \hat{D} of rank 3: For U_{xy} , $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_t = 0\}$, where $\varpi_t = dt - tadx - ady$ and a is a fiber coordinate. For U_{xt} , $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = 0\}$, where $\varpi_y = dy + tdx - bdt$ and b is a fiber coordinate. The decomposition $\Sigma(R) = \Sigma_0 \cup \Sigma_1$ is given by $\Sigma_0|_{p^{-1}(U)} = U_{xy}$, $\Sigma_1|_{p^{-1}(U)} = U_{xt} \setminus U_{xy}$, respectively.

By using the approach (i), we construct the geometric singular solutions of $(\Sigma(R), \hat{D})$ passing through Σ_1 . Let $\iota : S \hookrightarrow U_{xt}$ be a graph defined by

$$(x, y(x, t), z(x, t), p(x, t), q(x, t), t, b(x, t)).$$

If S is an integral submanifold of (U_{xt}, \hat{D}) , then the following conditions are satisfied:

$$(23) \quad \iota^* \varpi_0 = (z_x - p - qy_x)dx + (z_t - qy_t)dt = 0,$$

$$(24) \quad \iota^* \varpi_1 = \left(p_x - \frac{t^3}{3} - \frac{t^2}{2}y_x\right)dx + \left(p_t - \frac{t^2}{2}y_t\right)dt = 0,$$

$$(25) \quad \iota^* \varpi_2 = \left(q_x - \frac{t^2}{2} - ty_x\right)dx + (q_t - ty_t)dt = 0,$$

$$(26) \quad \iota^* \varpi_y = (y_x + t)dx + (y_t - b)dt = 0.$$

From these conditions, we have

$$(27) \quad z_x - p + qt = 0, \quad z_t - bq = 0,$$

$$(28) \quad p_x + \frac{t^3}{6} = 0, \quad p_t - \frac{bt^2}{2} = 0,$$

$$(29) \quad q_x + \frac{t^2}{2} = 0, \quad q_t - bt = 0,$$

$$(30) \quad y_x + t = 0, \quad y_t - b = 0.$$

We have $y = -tx + y_0(t)$ from (30). Note that the condition passing through Σ_1 is $y_t(0, 0) = y'_0(0) = 0$. From (29), we have $q = -t^2x/2 + ty_0(t) - \int y_0(t)dt$. From (28), we have $p = -t^3x/6 + t^2y_0(t)/2 - \int ty_0(t)dt$. From (27), we have

$$\frac{x^2t^3}{6} - x \left\{ \frac{t^2y_0(t)}{2} + \int ty_0(t)dt - t \int y_0(t)dt \right\} + \frac{ty_0^2(t)}{2} + \frac{1}{2} \int y_0^2(t)dt - y_0(t) \int y_0(t)dt.$$

Consequently, we obtain the singular solutions of the form

$$(31) \quad \begin{aligned} &(x, \quad -xt + y_0(t), \\ &\frac{x^2t^3}{6} - x \left\{ \frac{t^2y_0(t)}{2} + \int ty_0(t)dt - t \int y_0(t)dt \right\} + \frac{ty_0^2(t)}{2} + \frac{1}{2} \int y_0^2(t)dt - y_0(t) \int y_0(t)dt, \\ &-t^3x/6 + t^2y_0(t)/2 - \int ty_0(t)dt, \quad -t^2x/2 + ty_0(t) - \int y_0(t)dt, \quad t, \quad -x + y'_0(t)), \end{aligned}$$

where $y_0(t)$ is a function on S depending only t and which satisfies $y'_0(0) = 0$. From this condition $y'_0(0) = 0$, these solutions have singularities at the origin in $J^1(\mathbb{R}^2, \mathbb{R})$.

We next construct the singular solutions by using the approach (ii). From the structure equation (21), we have

$$\begin{aligned} Ch(D) &= \{\varpi_0 = \hat{\varpi}_1 = \varpi_2 = \omega_2 = \pi = 0\}, \\ &= \text{span} \left\{ \frac{\partial}{\partial x} - t \frac{\partial}{\partial y} + (p - tq) \frac{\partial}{\partial z} - \frac{t^3}{6} \frac{\partial}{\partial p} - \frac{t^2}{2} \frac{\partial}{\partial q} \right\}. \end{aligned}$$

Hence we have a local coordinate $(x_1, x_2, x_3, x_4, x_5)$ on the leaf space $B := R/Ch(D)$ given by

$$\begin{aligned} x_1 &:= z - xp + xqt + \frac{1}{6}x^2t^3, \quad x_2 := p - qt + \frac{1}{2}yt^2 + \frac{1}{6}t^3x, \\ x_3 &:= -q + \frac{1}{2}yt, \quad x_4 := y + xt, \quad x_5 := -t. \end{aligned}$$

Conversely, R is locally a \mathbb{R} -bundle on B . If we take a coordinate function λ of the fiber \mathbb{R} , then the coordinate (x, y, z, p, q, t) is expressed in terms of the coordinate

$(x_1, x_2, x_3, x_4, x_5, \lambda)$ defined by

$$(32) \quad \begin{aligned} x &= \lambda, & y &= x_4 + \lambda x_5, & z &= x_1 + \lambda x_2 - \frac{1}{2}\lambda x_4(x_5)^2 - \frac{1}{6}\lambda^2(x_5)^3, \\ p &= x_2 + x_3 x_5 + \frac{1}{6}\lambda(x_5)^3, & q &= -x_3 - \frac{1}{2}x_4 x_5 - \frac{1}{2}\lambda(x_5)^2, & t &= -x_5. \end{aligned}$$

On the base space B , we consider a rank 2 differential system $D_B = \{\alpha_1 = \alpha_2 = \alpha_3 = 0\}$ given by

$$\begin{aligned} \alpha_1 &= dx_1 + \left(x_3 + \frac{1}{2}x_4 x_5\right) dx_4, & \alpha_2 &= dx_2 + \left(x_3 - \frac{1}{2}x_4 x_5\right) dx_5, \\ \alpha_3 &= dx_3 + \frac{1}{2}(x_4 dx_5 - x_5 dx_4). \end{aligned}$$

It is well-known that this system D_B is a flat model of (2,3,5)-distributions [24]. Indeed, we can check this fact by calculating derived systems. Moreover, it is also well-known that D_B has infinitesimal automorphism G_2 ([2], [25]). For the projection $p : R \rightarrow B$, generator 1-forms of D and D_B are related as follows:

$$\varpi_0 := p^* \alpha_1 + x p^* \alpha_2, \quad \varpi_1 := p^* \alpha_2 - x p^* \alpha_3, \quad \varpi_2 := -p^* \alpha_3.$$

Thus, (B, D_B) is a retracting space of (R, D) , that is, $(B, D_B) = (p(R), p_* D)$. By using this correspondence, E. Cartan obtained the explicit description of regular solutions of (R, D) which are constructed by solution curves of (B, D_B) (cf. [2], [16]). In contrast to this result, we construct anew singular solutions in the following. We consider integral curves $c(\tau)$ of D_B given by

$$(33) \quad \begin{aligned} x_1 &= \int \left\{ \varphi' \int \varphi d\tau - \varphi \varphi' \tau \right\} d\tau, & x_2 &= \int \left\{ \int \varphi d\tau \right\} d\tau, \\ x_3 &= -\frac{1}{2} \int (\varphi - \tau \varphi') d\tau, & x_4 &= \varphi(\tau), & x_5 &= \tau. \end{aligned}$$

where τ is a parameter of curves, and $\varphi(\tau)$ is an arbitrary smooth function of τ . Here, we assume the condition $\varphi'(0) = 0$ to consider singular solutions which have singularities at the origin in $J^1(\mathbb{R}^2, \mathbb{R})$. Then, from relations (32) and (33), we obtain the following singular solutions:

$$(34) \quad \begin{aligned} &(x, \quad -xt + \varphi(-t), \quad \frac{x^2 t^3}{6} - x \left\{ \frac{t^2 \varphi(-t)}{2} + \int t \varphi(-t) dt - t \int \varphi(-t) dt \right\} \\ &+ \frac{t \varphi^2(-t)}{2} + \frac{1}{2} \int \varphi^2(-t) dt - \varphi(-t) \int \varphi(-t) dt, \\ &- \frac{t^3 x}{6} + \frac{t^2 \varphi(-t)}{2} - \int t \varphi(-t) dt, \quad -\frac{t^2 x}{2} + t \varphi(-t) - \int \varphi(-t) dt, \quad t). \end{aligned}$$

These singular solutions are equal to the singular solutions (31) obtained by the approach (i).

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